Complex Numbers

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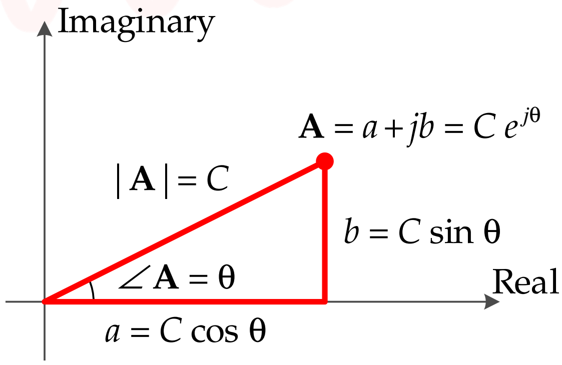
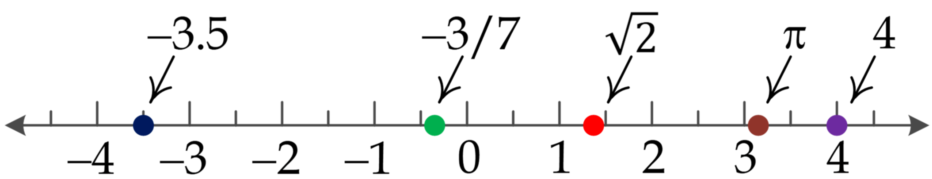


Real-life signals (manipulated by physical hardware) can be expressed using **real numbers**. We visualize the set of real numbers $\mathbb{R}$ using points on a one-dimensional horizontal axis



We can solve a wider range of mathematical problems (e.g., $x^2 = -1$) if we introduce the **imaginary unit**, or **imaginary operator**, denoted by i (or commonly in electrical engineering by j).

By definition,

$$j^2 = -1$$

Hence,

$$j = \sqrt{-1}$$

$$j^3 = j^2 \times j = -1 \times j \qquad j^3 = -j$$

$$j^4 = j^2 \times j^2 = -1 \times -1 \qquad j^4 = 1$$

$$j^5 = j$$

$$j^6 = -1$$

$$1/j = -j$$

The product of real number and the imaginary operator is called an **imaginary number** (e.g., $j5$ or $-j0.35$).

The sum of a real number and an imaginary number is called a **complex number**. Thus, a complex number is $\mathbf{A} = \vec{A} = \bar{A} = A = a + jb$, where a and b are real numbers.

The complex number **A** has a **real part** a and an **imaginary part** b:

$$\mathrm{Re}\{\mathbf{A}\} = a$$

$$\mathrm{Im}\{\mathbf{A}\} = b$$

Notice that the imaginary part of **A** is the real number b not the imaginary number jb. Hence, ***imaginary number*** (a complex number) should not be confused with its ***imaginary part*** (part of that number).

A real number is a special case of a complex number that has an imaginary part equal to zero (i.e., $a = a + j0$).

An imaginary number is a special case of a complex number that has a real part equal to zero (i.e., $jb = 0 + jb$).

The word complex means "composed of several parts".

The word imaginary is used because $j = \sqrt{-1}$ cannot be created in real-life.

The $a + jb$ form is called the **rectangular form** or rectangular representation or **Cartesian form** of the complex number. Other forms (exponential and polar forms) will be discussed later.

To visualize complex numbers graphically we use a rectangular (or Cartesian) coordinate system, with a horizontal axis (for real numbers) and a vertical axis (for imaginary numbers).

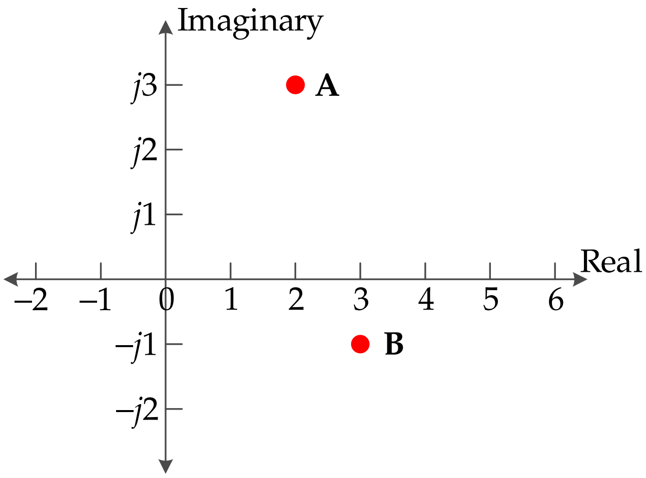
This is called a **complex plane**, or Argand diagram.

A complex number is represented in the complex plane as a single point.

Examples,

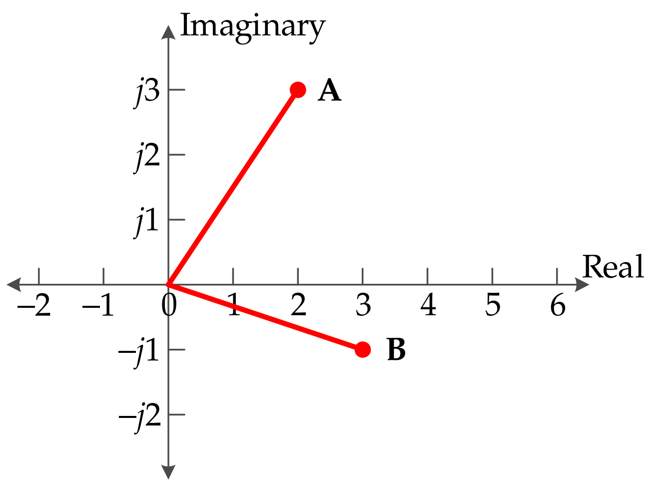
$$\mathbf{A} = 2 + j3$$

$$\mathbf{B} = 3 - j$$



Sometimes we represent the complex number as a line or vector, since its behavior reminds us of the behavior of a two-dimensional vector.

Complex numbers are **not** vectors, but they exhibit similar behavior to two-dimensional vectors, which makes memorizing their properties easier.



A complex number written as $\mathbf{A} = a + jb$ is said to be in the **rectangular** or **Cartesian form**.

Equality:

Two complex numbers are equal if, and only if, their real parts are equal and also their imaginary parts are equal.

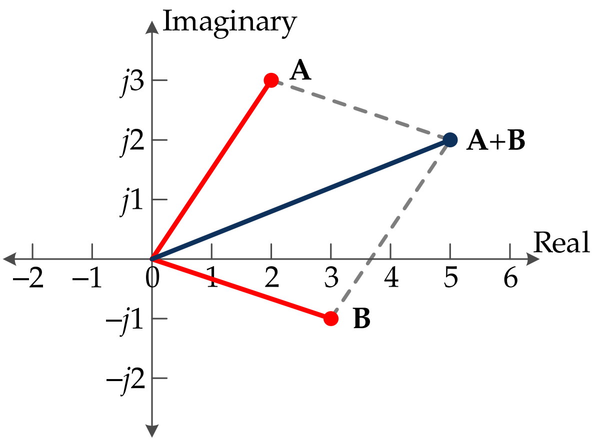
Addition:

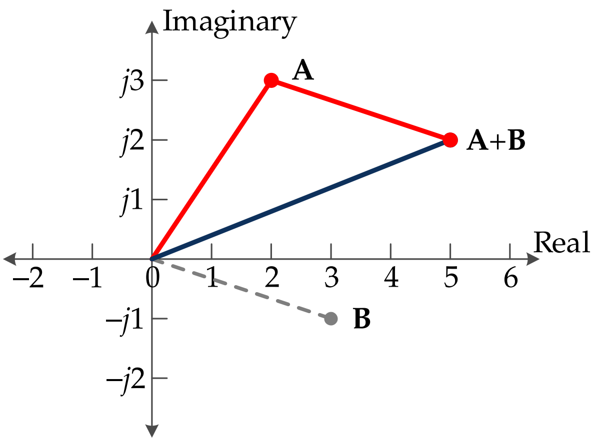
$$\mathbf{A} + \mathbf{B} = (a + jb) + (c + jd) = (a + c) + j(b + d)$$

Example,

$$\mathbf{A} + \mathbf{B} = (2 + j3) + (3 - j) = (2 + 3) + j\big(3 + (-1)\big) = 5 + j2$$

Similar to vectors, addition (and subtraction) of complex numbers can be done graphically using the complex plane, by completing the parallelogram, or by drawing the vectors in a head-to-tail shape.





Subtraction:

$$\mathbf{A} - \mathbf{B} = (a + jb) - (c + jd) = (a - c) + j(b - d)$$

Example,

$$\mathbf{A} + \mathbf{B} = (2 + j3) - (3 - j) = (2 - 3) + j\big(3 - (-1)\big) = -1 + j4$$

Multiplication: Notice there is no concept of dot versus cross product.

$$\mathbf{A}\,\mathbf{B} = \mathbf{A} \times \mathbf{B} = \mathbf{A} \cdot \mathbf{B} = (a + jb)(c + jd) = (ac - bd) + j(bc + ad)$$

Notice that $jb \times \ jd = j^2bd = -bd$.

Example:

$$\mathbf{A}\,\mathbf{B} = (2 + j3)(3 - j) = 6 - j^23 + j9 - j2 = (6 + 3) + j(9 - 2) = 9 + j7$$

Complex conjugate:

The complex conjugate of a complex number $\mathbf{A} = a + jb$ is $\mathbf{A}^* = a - jb$.

For example, if $\mathbf{A} = 2 + j3$, then its complex conjugate is $\mathbf{A}^* = 2 - j3$, and if $\mathbf{B} = 3 - j$, then its complex conjugate is $\mathbf{B}^* = 3 + j$.

Later on, we explain that this complex conjugate operation keeps the complex number magnitude as is, while changing the sign of its phase.

The complex conjugate for a full complex expression is obtained by replacing every complex term in the expression by its conjugate, or by replacing every j in the expression by $-j$.

Some useful properties of complex conjugate:

$$\mathbf{A} + \mathbf{A}^* = (a + jb) + (a - jb) = 2a = \text{real number}$$

$$\mathbf{A} - \mathbf{A}^* = (a + jb) - (a - jb) = j2b = \text{imaginary number}$$

$$\mathbf{A}\,\mathbf{A}^* = (a + jb)(a - jb) = a^2 + b^2 + jab - jab = a^2 + b^2 = \text{real number}$$

$$(\mathbf{A}^*)^* = (a - jb)^* = a + jb = \mathbf{A}$$

which also means that if $\mathbf{A}^*$ is the complex conjugate of $\mathbf{A}$, then $\mathbf{A}$ is the complex conjugate of $\mathbf{A}^*$.

Division:

Can multiply both the numerator and denominator by the complex conjugate of the denominator:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{a + jb}{c + jd} = \frac{\mathbf{A}\,\mathbf{B}^*}{\mathbf{B}\,\mathbf{B}^*} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{(c^2 + d^2)}$$

$$\frac{\mathbf{A}}{\mathbf{B}} = \left(\frac{ac + bd}{c^2 + d^2}\right) + j\left(\frac{bc - ad}{c^2 + d^2}\right)$$

Example:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{2 + j3}{3 - j} = \frac{(2 + j3)(3 + j)}{(3 - j)(3 + j)} = \frac{6 + j^2 3 + j9 + j2}{9 - j^2 + j3 - j3} = \frac{3 + j11}{9 + 1} = 0.3 + j1.1$$

Notice that the addition or subtraction of two complex numbers in rectangular form is easy to perform, but multiplication or division in rectangular form is a bit more difficult. If we write the complex numbers using the exponential or polar form, multiplication and division will be much simpler to perform (see later).

Reasons for defining and using complex numbers:

Complex numbers represent a valuable mathematical tool to simplify calculations in some areas of engineering analysis and/or design.

For example, in circuit analysis for alternating current (AC) systems:

Convert voltages & currents into phasors and convert resistance, inductance & capacitance into impedance (real numbers into complex numbers)

Set of differential equations → Set of linear equations (much easier to solve)

Convert the solution back to real numbers

The conversion plus solution plus going back is much simpler & faster than solving using real numbers.

Calculators and computer software support complex numbers.

Another example: Filter design, and communication system design:

Convert input/output signals & system response (real numbers) using Fourier transform into frequency domain (in general, a complex number)

Convolution → Multiplication (easier to solve & visualize)

Convert the solution back to real numbers using inverse Fourier transform

Euler's identity: You might see it in four different (yet equivalent) forms:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

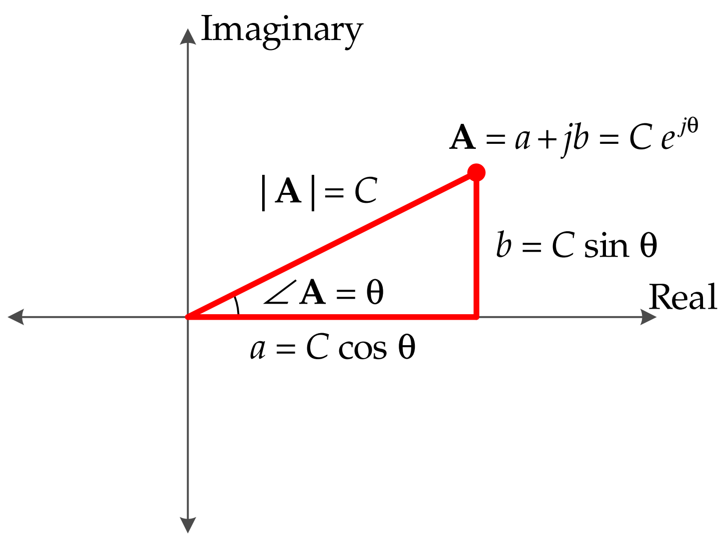
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Proof …

We can take advantage of Euler's identity to write complex numbers in the **exponential form**. This exponential form (which is equivalent to the **polar form**, to be explained shortly) says: Think about the complex number as a **magnitude** (or amplitude) and **phase** (or angle or argument) instead of a **real part** and an **imaginary part** (i.e., the rectangular form).



From Euler's identity, we can write a complex number

$$\mathbf{A} = a + jb = C\ e^{j\theta}$$

where:

$$a = C\cos\theta$$

$$b = C\sin\theta$$

$$C = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

You can easily see that this is true from the above triangle visualization of the complex number in the complex plane. Notice that C is positive.

Proof. Say we have a complex number written as

$$\mathbf{A} = C\ e^{j\theta}$$

Use Euler's identity, which says $e^{j\theta} = \cos\theta + j\sin\theta$ to get

$$\mathbf{A} = C\ e^{j\theta} = C\ (\cos\theta + j\sin\theta) = C\cos\theta\ + jC\sin\theta = a + jb$$

Hence, we get a complex number in the rectangular form with

$$a = C\cos\theta$$

$$b = C\sin\theta$$

To obtain C from a & b, square the above two equations and add them

$$a^2 + b^2 = C^2\cos^2\theta + C^2\sin^2\theta = C^2(\cos^2\theta + \sin^2\theta) = C^2$$

$$C = \sqrt{a^2 + b^2}$$

To obtain θ from a & b, divide the two equations and find the inverse

$$\frac{b}{a} = \frac{C\sin\theta}{C\cos\theta} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Example: Convert complex number $\mathbf{A} = \sqrt{12} + j2$ into exponential form. Solution:

$$C = \sqrt{a^2 + b^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2}{\sqrt{12}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \equiv 30°$$

$$\mathbf{A} = 4\,e^{j(\pi/6)} = 4\,e^{j30°}$$

For the phase (or angle), you have a choice between radians or degrees

$$\theta = \frac{\pi}{6} = \frac{\pi}{6}\text{ rad} = \frac{\pi}{6}\text{ radian} = 0.5236 = 0.5236\text{ rad} = 0.5236\text{ radian}$$

Or,

$$\theta = 30°$$

To convert from radians to degrees

$$\theta = \frac{\pi}{6} \implies \frac{\pi}{6} \times \frac{\mathbf{180°}}{\boldsymbol{\pi}} = \frac{180°}{6} = 30°$$

To convert from degrees to radians

$$\theta = 30° \implies 30° \times \frac{\boldsymbol{\pi}}{\mathbf{180°}} = \frac{\pi}{6} = 0.5236$$

I will mostly use radians, with occasional mention of degrees, but in other courses, degrees are used more often. The use of the degree symbol (°) avoids any possible confusion.

The transformation between the exponential (or polar) form and the rectangular form is available on most calculators. Just be careful to set your calculator to use either degrees or radians appropriately, as this is a common mistake that students make.

The is similar to the mistake that can be easily made when calculating $\cos\theta$, $\sin\theta$ or $\tan\theta$, without paying attention to whether the calculator is set to radians or degrees.

Convert the complex number $\mathbf{A} = 4\, e^{j(\pi/6)}$ into rectangular form

$$a = C\cos\theta = 4\cos\frac{\pi}{6} = 4 \times \frac{\sqrt{3}}{2} = \sqrt{12}$$

$$b = C\sin\theta = 4\sin\frac{\pi}{6} = 4 \times \frac{1}{2} = 2$$

$$\mathbf{A} = a + jb = \sqrt{12} + j2$$

Q1. Convert $\mathbf{B} = 4 + j3$ to exponential form.
Q1. Answer: $\mathbf{B} = 5\, e^{j0.6435}$.

Q2. Convert $\mathbf{D} = 2\, e^{j58^\circ}$ to rectangular form.
Q2. Answer: $\mathbf{D} = 1.0598 + j1.6961$.

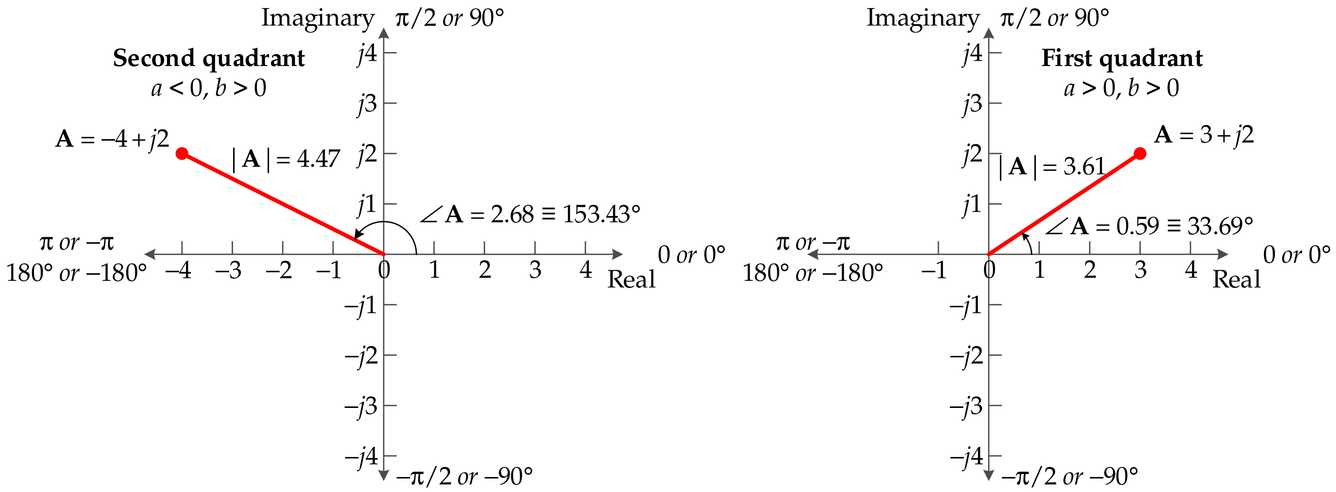
Be careful when converting from rectangular form to exponential (or polar) form and finding the phase (or angle) θ using:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

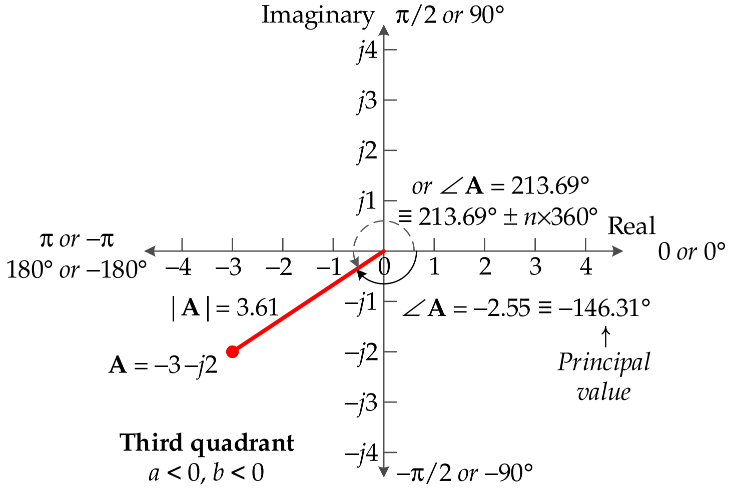
because the arctangent function is multivalued, and an appropriate angle must be selected from various possibilities. For example:

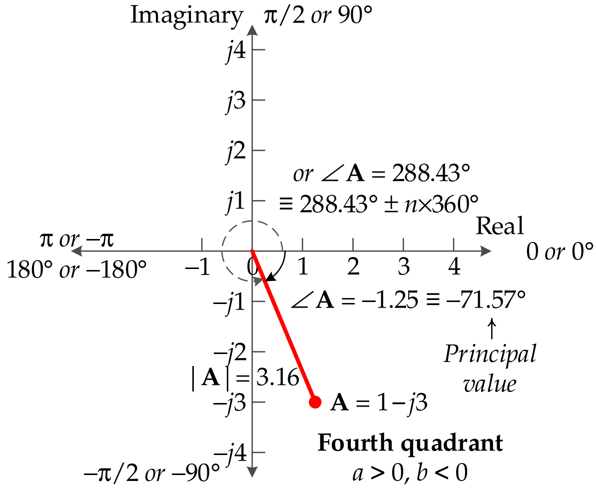
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° \text{ } or \text{ } 210° \text{ } or \text{ } 390° \text{ } or \text{ } 570° \text{ } or \text{ } ...$$

So, look at the signs of $a = C\cos\theta$ and $b = C\sin\theta$ in the rectangular form and select an angle θ within the proper quadrant that matches the signs of $\cos\theta$ and $\sin\theta$ (remember that C is always positive).



Notice that $\tan^{-1}(2/3) = 33.69°$ is possible and represents the **correct answer**, while $\tan^{-1}(2/3) = 213.69°$ is also possible but represents a wrong answer for $\mathbf{A} = 3 + j2$.





Some prefer to gives answers in the range $0° \leq \theta < 360°$, but I recommend using the principal value of the angle, which is $-180° < \theta \leq 180°$ or $-\pi < \theta \leq \pi$ whenever possible.

Typically, you can visualize the above sketches without the need to actually draw them on a piece of paper.

Note that electronic calculators typically produce an inverse tangent angle θ that is in the range $-90° < \theta < 90°$. For example, both $\tan^{-1}\big((-3)/4\big)$ and $\tan^{-1}\big(3/(-4)\big)$ is given by the calculator as $-36.87°$, which is not always what you want, since the first case is in the fourth quadrant, while the second case is in the second quadrant.

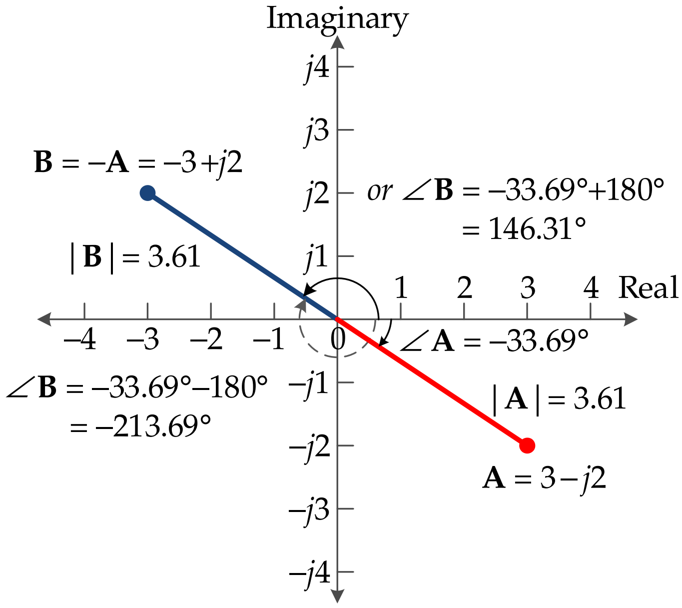
However, if you use the rectangular-to-polar conversion feature of your calculator, or your calculator supports complex numbers natively, you will get the correct angle (in the correct quadrant) for all cases.

Negative of a complex number

To get $-\mathbf{A}$ in rectangular form, negate both the real and imaginary parts of $\mathbf{A}$,

$$\mathbf{B} = -\mathbf{A} = -(a + jb) = -a - jb$$

For exponential (or polar form), keep the magnitude the same, but add or subtract 180° (or π radians) to the phase,



$$\mathbf{B} = -\mathbf{A} = -\left(C\ e^{j\theta}\right) = -C\ e^{j\theta} = C\ e^{j(\theta \pm \pi)}$$

Additionally, you can see from the complex plane that adding or subtracting 360° (or 2π radians) to the phase of a complex number does not affect it in any way.

Hence, two complex numbers, both written in exponential form, are equal if, and only if, their amplitudes are equal and their angles are the same or are equivalent. Equivalent angles are those which differ by multiples of 360° (or multiples of 2π radians). For example,

$$\left(\mathbf{A} = 5\ e^{j\pi/2}\right) = \left(\mathbf{B} = 5\ e^{j5\pi/2}\right) = \left(\mathbf{C} = 5\ e^{j-3\pi/2}\right)$$

Q3. Show that the complex conjugate of $\mathbf{A} = C\ e^{j\theta}$ is $\mathbf{A}^* = C\ e^{-j\theta}$.
Hint: Use Euler's identity.

The third form for representing a complex number is **polar form**, which is a *short hand notation* for the exponential form (it uses the same magnitude and the same phase as the exponential form). Hence, a complex number in the exponential form,

$$\mathbf{A} = 5\ e^{j35°}$$

is written in the polar form as,

$$\mathbf{A} = 5\ \angle 35°$$

The name comes from the ability to represent the complex number as a point in the polar coordinate system.

The transformation between rectangular and polar from is similar to the transformation between the rectangular and exponential form,

$$\mathbf{A} = 2 - j5 = 5.39\ e^{-j68.2°} = 5.39\ \angle{-68.2°}$$

Here is a summary of the conversion equations

$$\mathbf{A} = a + jb = \mathrm{Re}\{\mathbf{A}\} + j\mathrm{Im}\{\mathbf{A}\} = C\cos\theta + jC\sin\theta = C\ e^{j\theta}$$
$$= \sqrt{a^2 + b^2}\ e^{j\tan^{-1}\left(\frac{b}{a}\right)} = C\ \angle\theta = \sqrt{a^2 + b^2}\ \angle\tan^{-1}\left(\frac{b}{a}\right)$$

Many times, it is more intuitive to think about the magnitude and phase of a complex number rather than its real and imaginary parts. Also, multiplication and division are easier to perform using the exponential (or polar) forms.

Multiplication

Multiply the two magnitudes and add the individual angles.

Example:

$$\mathbf{A} = 2.6\ \angle 12° = 2.6\ e^{j12°}$$

$$\mathbf{B} = 5.2\ \angle{-65°} = 5.2\ e^{-j65°}$$

Solution:

$$\mathbf{A\ B} = \mathbf{A} \times \mathbf{B} = \mathbf{A} \cdot \mathbf{B} = \left(2.6\ e^{j12°}\right)\left(5.2\ e^{-j65°}\right) = 2.6 \times 5.2 \times e^{j(12°-65°)}$$

$$\mathbf{A\ B} = 13.52\ e^{-j53°} = 13.52\ \angle{-53°}$$

Division

Divide the two magnitudes and subtract the angles (no need to use the complex conjugate). Example:

$$\mathbf{A} = 2.6\ \angle 12° = 2.6\ e^{j12°}$$

$$\mathbf{B} = 5.2\ \angle{-65°} = 5.2\ e^{-j65°}$$

Solution:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{2.6\ e^{j12°}}{5.2\ e^{-j65°}} = \frac{2.6}{5.2} e^{j(12°+65°)} = 0.5\ e^{j77°} = 0.5\ \angle 77°$$

For addition and subtraction, it is easier to convert the complex numbers into rectangular form, add or subtract, then convert back.

Q4. Determine $\mathbf{B} = (2.6\ \angle 12°)^4$

Q4. Solution. The polar form is similar to the exponential form,

$$\mathbf{B} = \left(2.6\ e^{j12°}\right)^4 = (2.6)^4\left(\ e^{j12°}\right)^4 = (2.6)^4\ e^{j4\times 12°}$$

$$\mathbf{B} = 45.7\ e^{j48°} = 45.7\ \angle 48°$$

Q5. Determine $\mathbf{M} = \ln(5.2\ \angle{-65°})$

Q5. Solution. Using the exponential form,

$$\mathbf{M} = \ln(5.2\ \angle{-65°}) = \ln\left(5.2\ e^{-j65°}\right) = \ln(5.2) + \ln\left(e^{-j65°}\right)$$

$$\mathbf{M} = \ln(5.2) + -j65° \times \frac{\pi}{180°} = 1.6487 - j1.1345$$

$$\mathbf{M} = 2.0\ \angle{-34.53°}$$